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1996 J. Phys. A: Math. Gen. 29 4967

(http://iopscience.iop.org/0305-4470/29/16/018)

J. Phys. A: Math. Gen. **29** (1996) 4967–4975. Printed in the UK

Singular travelling wave solutions of the fifth-order KdV, Sawada-Kotera and Kaup equations

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Received 25 July 1995, in final form 6 March 1996

Abstract. We obtain second-order equations of degree four (six), for travelling wave solutions of the KdV (Sawada-Kotera/Kaup) equations, which reduce to first-order equations for monotone solitary waves. For the KdV equation, the singular solutions of this equation with an asymptotic value b consist of the well known sech^2 solution and a new solution with a non-zero asymptotic value depending on the wave speed. We show that the well known solitary wave solutions are determined uniquely as the singular solutions with asymptotic value $b = 0$, which are also stationary with respect to the wave speed.

1. Introduction

Evolution equations of the form

$$u_t = \frac{\partial}{\partial x}[(\alpha u_4 + \beta u u_2 + \gamma u_1^2 + r u^3) + (\mu u_2 + q u^2) + (pu)] \tag{1.1}$$

where $u_k = \partial^k u/\partial x^k$ are proposed as model equations for water waves [1]. The 'travelling wave' solutions are of the form $u(x,t) = u(x - ct)$, where c is the wave speed. As we are concerned with travelling waves, we redefine $c \to (c + p)$, hence set $p = 0$. Furthermore we take $\alpha = \beta = 1$ by rescaling u and x. These equations are integrable only for the following values of the parameters:

5th order KdV equation (KdV5): $\gamma = \frac{1}{2}$ $\quad r = \frac{1}{10}$ $\quad q = \frac{3}{10}\mu$

Sawada-Kotera equation: $\gamma = 0$ $\quad r = \frac{1}{15}$ $\quad \mu = q = 0$

Kaup equation: $\gamma = \frac{3}{4}$ $\quad r = \frac{1}{15}$ $\quad \mu = q = 0.$

These integrable equations with $u(x,t) = u(x - ct)$ admit first integrals that can be obtained using conserved covariants. It turns out that only the first three of these first integrals (equations (2.1) and (2.2)) are functionally independent and their existence leads to the classification above. Thus the differential equations for travelling waves can be reduced to second-order ordinary differential equations. By eliminating u_4 and u_3 among these equations, we obtain second-order equations which are of degrees 4 and 6, respectively, for the KdV5 and Sawada-Kotera/Kaup equations (equations (2.7) and (2.8)). The non-constant factors appearing in the elimination process, except for equation (2.6), are not solutions of the original equations. This non-constant factor gives a periodic solution of the Kaup equation.

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0305-4470/96/164967+09$19.50

Monotone symmetric travelling waves are described by the relation $u_1^2 = W(u)$, where W is positive for $u \in (b, a)$, and $W(a) = W(b) = 0$. If furthermore $W'(b) = 0$, then the corresponding wave has an asymptotic value at $u = b$, else it is a periodic solution. Since $u_2 = \frac{1}{2}W'$, the equations for monotone symmetric travelling waves reduce to first-order equations of the form $F(W', W, u, c) = 0$. In sections 3, 4 and 5 we study, respectively, the singular solutions of the KdV5, Sawada-Kotera and Kaup equations.

For the KdV5 equation we consider solutions that have an asymptotic value b not necessarily zero. We show that there is a unique envelope corresponding to the well known sech^2 solution. The other singular solution gives a new solution whose asymptotic value and maximum amplitude depend on the wave speed. We also show that the sech^2 solution is stationary with respect to the wave speed, i.e. $F_c = 0$.

For the Sawada-Kotera and Kaup equations, because of computational limitations, we consider only those solutions where all the integration constants are zero. In this case we also obtain some new solutions, and we show that the usual solitary wave solutions are uniquely determined as those singular solutions that are also stationary with respect to the wave speed.

2. Equations for travelling waves

2.1. Integrations using conserved covariants

We recall certain concepts. A differential function $F[u]$ is a differentiable function of the derivatives of u up an arbitrary but fixed order. On the space of differential functions, an inner product is defined by $\langle f, g\rangle = \int_{-\infty}^{+\infty} (fg)\,\mathrm{d}x$. Adjoints of linear operators are defined via this inner product.

A differential function σ is called a *symmetry* if it satisfies the linearized equation $\sigma_t - F_*\sigma = 0$, where $F_* = \sum_{j=0}^{j=N} \frac{\partial F}{\partial u_j} D^j$, if $F = F(u, u_1, \ldots, u_N)$. A differential function γ is called a *conserved covariant* if it satisfies the equation $\gamma_t + F_*^\dagger \gamma = 0$. A *recursion operator* R is a linear operator that sends symmetries to symmetries. The adjoint of the recursion operator $R^\dagger$ sends conserved covariants to conserved covariants. In the terminology of [3] integrable equations have an infinite number of symmetries, hence conserved covariants.

For the KdV5, Sawada-Kotera and Kaup equations it can easily be seen that $\gamma_0 = 1$ is a conserved covariant and it is known that all higher symmetries are total derivatives. Using these properties one can obtain an infinite number of first integrals as follows.

Let $\sigma_0 = F[u]$, $\gamma_n = (R^\dagger)^n\gamma_0$ and $\sigma_n = R^n\sigma_0$. Then

$$\int \gamma_n F = \langle\gamma_n, \sigma_0\rangle = \langle R^{\dagger n}\gamma_0, \sigma_0\rangle = \langle\gamma_0, R^n\sigma_0\rangle = \langle\gamma_0, \sigma_n\rangle.$$

But the second inner product is just the integral of σ_n and since all symmetries are total derivatives, $(\gamma_n F)$ is a total derivative for all n. Since the time derivative can always be incorporated into a u_1 term, we obtain an infinite number of first integrals of (1.1) with the additional restriction $u(x, t) = u(x - ct)$. It has been observed that only the first three of these first integrals (for travelling wave solutions) are functionally independent: these are the ones obtained exactly by the conserved covariants that give a classification of the equations. In the following computations, we take $c \neq 0$. The computations of conserved covariants are done using an integration package [4] with the symbolic programming language REDUCE.

The computation of the first integrals I_0, I_1, I_2 is straightforward; the results are given below.

KdV5 equation:

Conserved covariants: $\gamma_0 = 1 \qquad \gamma_1 = u \qquad \gamma_2 = u_2 + \frac{3}{10}u^2.$

First integrals:

$$I_0 = u_4 + u_2 u + u_2\mu + \tfrac{1}{2}u_1^2 + \tfrac{1}{10}u^3 + \tfrac{3}{10}u^2\mu + uc + d_0 \tag{2.1a}$$

$$I_1 = u_4 u - u_3 u_1 + \tfrac{1}{2}u_2^2 + u_2 u^2 + u_2 u\mu - \tfrac{1}{2}u_1^2\mu + \tfrac{3}{40}u^4 + \tfrac{1}{5}u^3\mu + \tfrac{1}{2}u^2 c + d_1 \tag{2.1b}$$

$$\begin{aligned} I_2 = u_4 u_2 &+ \tfrac{3}{10}u_4 u^2 - \tfrac{1}{2}u_3^2 - \tfrac{3}{5}u_3 u_1 u + \tfrac{4}{5}u_2^2 u + \tfrac{1}{2}u_2^2\mu + \tfrac{3}{5}u_2 u_1^2 \\ &+ \tfrac{3}{10}u_2 u^3 + \tfrac{3}{10}u_2 u^2\mu + \tfrac{1}{2}u_1^2 c + \tfrac{9}{500}u^5 + \tfrac{9}{200}u^4\mu + \tfrac{1}{10}u^3 c + d_2. \end{aligned} \tag{2.1c}$$

Sawada-Kotera/Kaup equations:

Conserved covariants: $\gamma_0 = 1 \qquad \gamma_1 = u_2 + \left(\frac{2}{5}\gamma + \frac{1}{10}\right)u^2$
$\gamma_2 = u_4 + \left(\frac{4}{5}\gamma + \frac{3}{5}\right)uu_2 + \left(\frac{2}{5}\gamma + \frac{3}{10}\right)u_1^2 + \left(-\frac{4}{25}\gamma^2 + \frac{17}{75}\gamma + \frac{2}{75}\right)u^3.$

First integrals:

$$I_0 = u_4 + u_2 u + u_1^2\gamma + \tfrac{1}{15}u^3 + uc + d_0 \tag{2.2a}$$

$$\begin{aligned} I_1 = u_4 u_2 &+ u_4 u^2\left(\tfrac{2}{5}\gamma + \tfrac{1}{10}\right) - \tfrac{1}{2}u_3^2 + u_3 u_1 u\left(-\tfrac{4}{5}\gamma - \tfrac{1}{5}\right) + u_2^2 u\left(\tfrac{2}{5}\gamma + \tfrac{3}{5}\right) + u_2 u_1^2\left(\tfrac{4}{5}\gamma + \tfrac{1}{5}\right) \\ &+ u_2 u^3\left(\tfrac{2}{5}\gamma + \tfrac{1}{10}\right) + \tfrac{1}{2}u_1^2 c + u^5\left(\tfrac{2}{125}\gamma + \tfrac{1}{250}\right) + u^3 c\left(\tfrac{2}{15}\gamma + \tfrac{1}{30}\right) + d_1 \end{aligned} \tag{2.2b}$$

$$\begin{aligned} I_2 = \tfrac{1}{2}u_4^2 &+ u_4 u_2 u\left(\tfrac{4}{5}\gamma + \tfrac{3}{5}\right) + u_4 u_1^2\left(\tfrac{2}{5}\gamma + \tfrac{3}{10}\right) + u_4 u^3\left(\tfrac{8}{75}\gamma + \tfrac{2}{75}\right) + u_3^2 u\left(-\tfrac{2}{5}\gamma + \tfrac{1}{5}\right) \\ &+ u_3 u_2 u_1\left(\tfrac{2}{5}\gamma - \tfrac{1}{5}\right) + u_3 u_1 u^2\left(-\tfrac{8}{25}\gamma + \tfrac{3}{25}\right) + u_3 u_1 c + u_2^3\left(-\tfrac{2}{5}\gamma + \tfrac{1}{15}\right) \\ &+ u_2^2 u^2\left(\tfrac{14}{25}\gamma + \tfrac{6}{25}\right) - \tfrac{1}{2}u_2^2 c + u_2 u_1^2 u\left(\tfrac{26}{25}\gamma + \tfrac{3}{50}\right) + u_2 u^4\left(\tfrac{8}{75}\gamma + \tfrac{2}{75}\right) \\ &+ u_1^4\left(\tfrac{7}{50}\gamma + \tfrac{3}{50}\right) + u_1^2 u^3\left(\tfrac{2}{75}\gamma + \tfrac{1}{50}\right) + u_1^2 uc\left(\tfrac{2}{5}\gamma + \tfrac{3}{10}\right) \\ &+ u^6\left(\tfrac{4}{1125}\gamma + \tfrac{1}{1125}\right) + u^4 c\left(\tfrac{2}{75}\gamma + \tfrac{1}{150}\right) + d_2. \end{aligned} \tag{2.2c}$$

Remark 1. If $u = b$ is an asymptotic value, then $u_i|_{u=b} = 0$ for $i > 0$. Substituting these in the first integrals we can obtain the expressions of d_i, $i = 0, 1, 2$ in terms of b. If there are two distinct asymptotic values b and b', then we should have $d_i(b) = d_i(b')$, but it is checked that these equations have no consistent solution $b \neq b'$, hence the KdV5, Sawada-Kotera and Kaup equations cannot have kink solutions.

Remark 2. If $u = a$ is an extremum value, then $u_{2k+1}|_{u=a} = 0$. This condition gives the allowable relationships among the amplitude and the speed of travelling waves, as below. In the following, we use the notation $u_k|_{u=a} = a_k$.

KdV5 equation with asymptotic value b:

(i) $50c - 4a^2 - 10ab - b^2 = 0 \qquad \mu = -\frac{3}{5}a - \frac{2}{5}b$

$$a_2 = \pm\tfrac{1}{10}(a-b)^2 \qquad a_4 = \mp\tfrac{1}{25}(a-b)^3 \tag{2.3a}$$

(ii) $2a^2 + 2a(3b + 5\mu) + 7b^2 + 20b\mu + 50c = 0$

$$a_2 = -\tfrac{1}{10}(a-b)^2 \qquad a_4 = \tfrac{1}{25}(a-b)^3 \tag{2.3b}$$

(iii) $a^2 - 2ab + 6b^2 + 10b\mu + 100c - 25\mu^2$

$a_2 = -\frac{1}{10}[2a^2 + a(b + 5\mu) - b(3b + 5\mu)]$

$$a_4 = \tfrac{1}{100}(a-b)[11a^2 + 2a(9b + 20\mu) - 4b^2 + 10b\mu + 25\mu^2]. \tag{2.3c}$$

Sawada-Kotera equation with $d_0 = d_1 = d_2 = 0$:

$$225c + 4a^2 = 0 \qquad a_2 = -\tfrac{1}{15}a^2 \qquad a_4 = \tfrac{4}{225}a^3. \tag{2.4}$$

Kaup equation with $d_0 = d_1 = d_2 = 0$:

$$\text{(i)}\quad 225c + a^2 = 0 \qquad a_4 = \tfrac{46}{225}a^3 \qquad a_2 = -\tfrac{4}{15}a^2 \tag{2.5a}$$

$$\text{(ii)}\quad 25c + a^2 = 0 \qquad a_4 = -\tfrac{2}{75}a^3 \qquad a_2 = 0 \tag{2.5b}$$

$$\text{(iii)}\quad 100c + a^2 = 0 \qquad a_4 = \tfrac{73}{300}a^3 \qquad a_2 = -\tfrac{3}{10}a^2. \tag{2.5c}$$

We note that (2.3*b*) reduces to (2.3*a*) for the specific value of μ. This solution is obtained as a special case, by the vanishing of a coefficient during the elimination process. Equations (2.3*b*), (2.4) and (2.5*b*) correspond to the usual solitary waves, while equation (2.3*c*) corresponds to the new solution given in section 4.

For the Sawada-Kotera and Kaup equations, the list of possible maximum amplitudes for a non-zero asymptotic value is quite large, but as the singular solutions are studied only for the case $d_0 = d_1 = d_2 = 0$, these results are also omitted.

2.2. Second-order nonlinear equations for travelling waves

The elimination of higher derivatives from the first integrals (2.1)–(2.2) leads to second-order equations for u. During this elimination process the equations obtained by the vanishing of the coefficients of higher-order derivatives also give certain solutions. Such a case occurs only for the Kaup equation.

A solution of the Kaup equation:

$$u_1\big(u_2 + \tfrac{2}{5}u^2 + 10c\big) = 0. \tag{2.6}$$

Assuming $u_1 \neq 0$ and integrating we obtain $u_1^2 = -\frac{4}{15}u^3 - 20uc + 20d_0$, and from the first integrals we get $d_1 = -10cd_0$, $d_2 = \frac{50}{3}c^3 - 2d_0^2$. It can be seen that only static solutions can have an asymptotic value. Since u_1^2 is equal to a third-order polynomial with distinct roots, the solutions are periodic functions given in terms of elliptic integrals [5].

The eliminations lead to the following equations. We omit the explicit expression of the coefficients.

KdV5 equation:

$$u_2^4 + [A_{21}u_1^2 + A_{22}]u_2^2 + [A_{11}u_1^4 + A_{12}u_1^2]u_2 + [A_{01}u_1^4 + A_{02}u_1^2 + A_{03}] = 0. \tag{2.7}$$

Sawada-Kotera/Kaup equations:

$$\begin{aligned} 0 = u_2^6 &+ A_5u_2^5 + [A_{41}u_1^2 + A_{42}]u_2^4 + [A_{31}u_1^4 + A_{32}u_1^2 + A_{33}]u_2^3 \\ &+ [A_{21}u_1^4 + A_{22}u_1^2 + A_{23}]u_2^2 + [A_{11}u_1^6 + A_{12}u_1^4 + A_{13}u_1^2]u_2 \\ &+ [A_{01}u_1^8 + A_{02}u_1^6 + A_{03}u_1^4 + A_{04}u_1^2 + A_{05}]. \end{aligned} \tag{2.8}$$

2.3. Monotone symmetric travelling waves

Monotone symmetric travelling waves can be described by the equation $u_1^2 = W(u)$ where $W(u)$ is positive for $b < u < a$ and $W(b) = W(a) = 0$ [1]. We recall that if $u_1 = \mathrm{d}u/\mathrm{d}x = \sqrt{W(u)}$, then $x(u)$ can be obtained by integrating $W^{-1/2}$. Precisely, if $u_0 \in (b, a)$, then $x(u) = \int_{u_0}^{u} W^{-1/2}(\xi)\,\mathrm{d}\xi$ for $u > u_0$, and $x(u) = -\int_{u_0}^{u} W^{-1/2}(\xi)\,\mathrm{d}\xi$ for $u < u_0$. As $x(u)$ is an increasing function, its inverse $u(x)$ is defined and it is also increasing. If the second integral is divergent then the domain of definition of W is semi-infinite. If W vanishes at an end point where it is differentiable, then all the odd derivatives of u vanish there, hence u can be extended as a symmetric function. Thus in particular, if

W is analytic, and has a simple zero at $u = a$ and a double zero at $u = b$, then it leads to a monotone symmetric wave with asymptotic value b and maximum amplitude a.

Substituting $u_1^2 = W$, $u_2 = \frac{1}{2}W'$ in equations (2.7)–(2.8), we obtain first-order equations $F(W', W, u, c, b) = 0$ for W. For the KdV5 case, as (2.7) is a quartic in W', the general solution can in principle be obtained as an algebraic function. However, the determination of the properties such as the number of real branches and the differentiability of the corresponding equations is difficult to investigate.

We give here a brief review of the properties of solutions of nonlinear equations of the form $F(W', W, u, c) = 0$, based on [2, section 3.5]. Let F be polynomial in W' and single-valued in W and u. If $F = 0$ has a simple root W_0' when $W = W_0$, $u = u_0$, then W' can be solved from $F = 0$ as a single-valued function $W' = f(W, u, c)$ in a neighbourhood of (u_0, W_0, W_0'). Then there is a unique solution with the initial conditions $W' = W_0'$, $W = W_0$ and $u = u_0$, provided that $f(W, u, c)$ satisfies a Lipschitz condition. However, even in the KdV5 case where we can obtain the functions $f(W, u, c)$, it is not possible to check the Lipschitz condition, hence to discuss uniqueness of solutions (for each of the branches). On the other hand, at points (W_0', W_0, u_0) where $F(W', W, u, c) = 0$ has a repeated root, the existence theorems are not applicable at all, but the study of these singular points gives important information about the structure of the solutions, the most important being the fact that the envelope of solutions, if it exists, is a singular solution that appears as a squared factor in the p-discriminant. We report below some definitions and certain results in [2, section 3.5].

The triad (W_0', W_0, u_0) for which

$$F(W', W, u, c) = 0 \qquad F_{W'}(W', W, u, c) = 0 \tag{2.9}$$

is called a *singular line element*. At such a point the equation $F(W_0', W_0, u_0, c) = 0$ has a multiple root W_0', hence such points are called *multiple points*. The polynomial obtained from the elimination of W' among these equations is called the *p-discriminant locus*. The factors of this polynomial describe various branches of the p-discriminant locus. These branches may or may not be solutions of the differential equation $F(W', W, u, c) = 0$. The p-discriminant locus may contain (i) the envelope of solutions (if there is an envelope), (ii) the locus of multiple points with coincident tangents (such as cusps), (iii) the *tac-locus* which consists of the points where different integral curves have a common tangent, (iv) the particular solutions corresponding to a specific value of the integration constant in the general solution. A branch of the p-discriminant is a polynomial $P(W, u, c) = 0$. One can check directly whether this polynomial actually gives a solution of the original equation, in which case we obtain a *singular solution*. If $P(W, u, c)$ is a squared factor of the p-discriminant and it gives rise to a solution, this solution is an envelope, else it gives the tac-locus.

We will study the p-discriminant locus for the KdV5 for arbitrary asymptotic value b, but we take $b = 0$ for the Sawada-Kotera and Kaup equations due to computational limitations.

Critical points of the KdV5 equation:

$$25W^2 + 10u^3W + 25u^2W\mu + u^6 + 5u^5\mu + 25u^4c = 0. \tag{2.10}$$

Critical points of the Sawada-Kotera equation:

$$225W^2 + 60Wu^3 + 4u^6 + 225u^4c = 0. \tag{2.11}$$

Critical points of the Kaup equation ($c = -k^2$):

$$225W^2 + 120W(u + 5k)^2(u - 10k) + 16u^2(u + 5k)^3(u - 15k) = 0 \tag{2.12a}$$

$$225W^2 + 120Wu(u^2 - 75k^2) + 16(u^2 - 25k^2)^2(u^2 - 100k^2) = 0. \tag{2.12b}$$

3. Singular solutions of the KdV5 equation with an asymptotic value b

The equation for monotone symmetric travelling wave solutions of the KdV5 equation is a fourth degree polynomial in W' of the form $F(W', W, u, c, b, \mu) = 0$. The singular solutions are obtained by eliminating W' among $F = 0$ and $F_{W'} = 0$, factorizing the resulting expression, and selecting those factors $P(W, u, c, b, \mu)$ such that when W' is solved from $\mathrm{d}P/\mathrm{d}u = 0$, it satisfies $F = 0$ modulo $P = 0$. With this method we obtain the following singular solutions.

$$W^2 + \tfrac{1}{5}W(-u+b)^2(2u+3b+5\mu) + \tfrac{1}{50}(-u+b)^4[2u^2+2u(3b+5\mu)+7b^2+20b\mu+50c] = 0 \qquad (3.1a)$$

$$W^2 + \tfrac{2}{135}W(2u+3b+5\mu)[13u^2+2u(-3b+10\mu)+63b^2+120b\mu+900c-200\mu^2] + \tfrac{1}{27}[u^2+2u(b+2\mu)+3b^2+8b\mu+20c]^2 \times[u^2-2ub+6b^2+10b\mu+100c-25\mu^2] = 0. \qquad (3.1b)$$

Equation (3.1a) leads to the usual sech^2 solution, and it is actually a squared factor of the p-discriminant, hence it is the envelope of solutions with an asymptotic value b.

Equation (3.1b) is a simple factor of the p-discriminant, thus it is a solution corresponding to a special value of the integration constant in the general solution. We will show that when c and b belong to a certain open set, there are solutions $W(u)$ leading to monotone symmetric waves of elevation whose asymptotic values and maximum amplitudes both depend on the wave speed.

We make the following change of variables

$$u = v + b \qquad \mu' = b + \mu \qquad c' = 10c - 3\mu^2. \qquad (3.2)$$

Then equation (3.1b) can be solved as

$$W^{\pm} = -\tfrac{1}{135}(2v+5\mu')(13v^2+20v\mu'+70\mu'^2+90c') \pm \tfrac{1}{135}(v^2+20v\mu'+10\mu'^2-30c')^{3/2}. \qquad (3.3)$$

Defining

$$z = \frac{v}{\mu'} \qquad \xi = \frac{c'}{\mu'^2} \qquad (3.4)$$

we obtain

$$\mu'^3 W^{\pm} = -\tfrac{1}{135}(2z+5)(13z^2+20z+70+90\xi) \pm \tfrac{1}{135}(z^2+20z+10-30\xi)^{3/2}. \qquad (3.5)$$

We will show that for $\xi < -3$, W^+ have a double zero at $b(\xi)$ and a simple zero at $a(\xi)$, with $W^+ > 0$ for $b(\xi) < z < a(\xi)$.

Let

$$\begin{aligned} q_1 &= z^2+20z+10-30\xi \\ q_2 &= z^2+5+10\xi \\ q_3 &= 13z^2+20z+70+90\xi \\ q_4 &= z^2+4z+6+2\xi \\ q_5 &= 2z+5. \end{aligned} \qquad (3.6)$$

The zero sets of the q_i, $i = 1, \ldots, 5$ are shown in figure 1. It can be checked that the only intersection point of these curves is $z = -\frac{5}{2}$, $\xi = -\frac{9}{8}$.

For definiteness we work with W^+ that leads to a wave of elevation. By similar considerations, W^- leads to a wave of depression. First note that since $135\mu'^3W^+ = -q_5q_3 + q_1^{3/2}$, W^+ is real for $q_1 > 0$, i.e. except for regions (1) and (10) in figure 1.

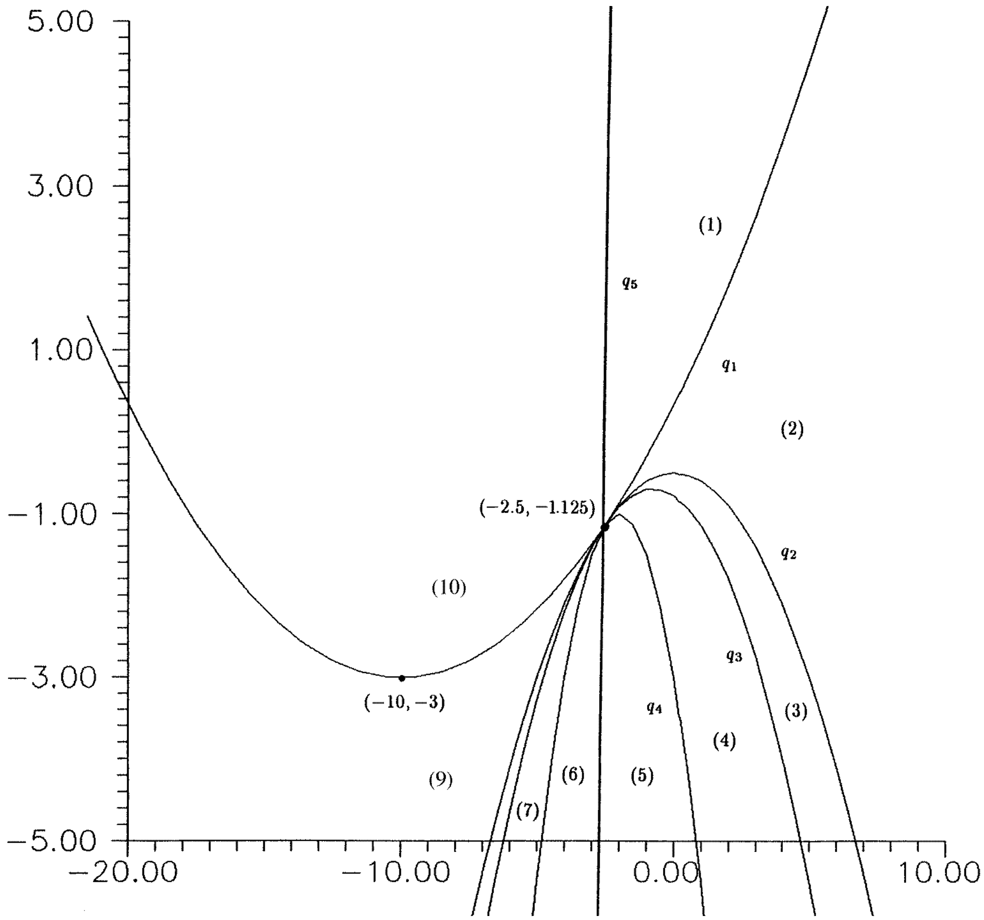

Figure 1.

Let B and C be defined by $B = q_5 q_3$ and $q_1^3 = B^2 - 4C$. Then one can compute $C = q_2 q_4^2$. As the square root is always positive, W^+ can have a zero only when $B > 0$ and $C = 0$. The regions where $B > 0$ are (2)–(3) and (6)–(7) in figure 1. On the other hand C can vanish when either q_2 or q_4 vanishes. Thus a zero of W^+ can occur either at the boundary of the regions (2)–(3) or (6)–(7). On the boundary of (2)–(3), $q_2 = 0$, and C changes sign in passing from that boundary. On the other hand, on the boundary of (6)–(7), $q_4 = 0$, and W^+ does not change sign, hence it has a double zero. Therefore, if a line $\xi =$ constant intersects the curves q_4 and q_2 without intersecting q_1 we have a real solution with a double zero $b(\xi)$ and a simple zero $a(\xi)$.

4. Singular solutions of the Sawada-Kotera equation for $d_0 = d_1 = d_2 = 0$

The singular solutions of the Sawada-Kotera equation for $d_0 = d_1 = d_2 = 0$ are obtained again by eliminating W' among $F = 0$ and $F_{W'} = 0$, and selecting the factors that are actually solutions of $F = 0$. The results are given below.

$$225W^2 + 60 * Wu^3 + 4u^6 + 225u^4 c = 0 \tag{4.1a}$$

$$W^3 + \left(\tfrac{1}{3}u^3 + 15uc\right)W^2 + \left(\tfrac{8}{225}u^6 - \tfrac{4}{3}u^4c - \tfrac{100}{3}u^2c^2 + \tfrac{2500}{9}c^3\right)W + \tfrac{4}{3375}u(u^4 - 75u^2c + 1875c^2)^2 = 0. \tag{4.1b}$$

The first solution leads to the usual sech^2 solution. It is a squared factor, hence an envelope, and it is also a solution which is stationary with respect to the wave speed.

In the second case, $W(u)$ is a solution of the differential equation $F = 0$, but it does not lead to a real solution $u(x)$ for the Sawada-Kotera equation. Because of this, we recall that we need W to be a function which is positive between two zeros at $u = b$ and $u = a$, in order to have a real wave of elevation. However, equation (4.1b) with $W = 0$ has no real solutions for u, hence equation (4.1b) does not lead to a real solution for u.

5. Singular solutions of the Kaup equation for $d_0 = d_1 = d_2 = 0$

The singular solutions of the Kaup equation obtained as described above are given below. In this case, it is convenient to set $c = -k^2$.

$$W + \tfrac{3}{10}u^3 - 30uk^2 = 0 \tag{5.1a}$$

$$W + \tfrac{4}{5}u^3 - 20uk^2 = 0 \tag{5.1b}$$

$$W + \tfrac{4}{15}u^3 - 20uk^2 \mp \tfrac{200}{3}k^3 = 0 \tag{5.1c}$$

$$W_{1,2} = -\tfrac{4}{15}u^3 + 20uk^2 + \tfrac{200}{3}k^3 \pm \sqrt{25k^2 + 5ku}\left(\tfrac{8}{3}uk + \tfrac{40}{3}k^2\right) \tag{5.1d}$$

$$W_{3,4} = -\tfrac{4}{15}u^3 + 20uk^2 - \tfrac{200}{3}k^3 \pm \sqrt{25k^2 - 5ku}\left(-\tfrac{8}{3}uk + \tfrac{40}{3}k^2\right). \tag{5.1e}$$

Among the solutions above, equations (5.1a–b) have three simple zeros, hence they can be solved in terms of elliptic integrals and lead to periodic solutions.

The two solutions given by equation (5.1c) have a double root at $u = \mp 5k$ and a simple root at $u = \pm 10k$. In the first case (with a double root at $-5k$), W is positive for $u \in (-5k, 10k)$, hence it gives a wave of elevation in the form of a sech^2 wave.

The last four solutions given by equations (5.1d–e) have, respectively, double and simple roots for the values $(5k, 0)$, $(5k, -15k)$, $(-5k, 15k)$, $(-5k, 0)$. Here a double root means a point where both W and W' vanish. We note that in all cases W is only C^1 at the double root. The solutions with a simple zero (hence maximum amplitude) at $u = 0$ lead to the well known 'anomalous' soliton solution of the Kaup equation. On the other hand, the pair with simple zeros at $\pm 15k$ are negative in between that asymptotic value and the maximum amplitude. Thus they can be considered as leading to solutions of the Kaup equation if we allow x to be complex variable.

The 'anomalous' solitary wave solution of the Kaup equation [1] is

$$u(\alpha(x - ct)) = 15\alpha^2 \frac{1 + 2\cosh(\alpha(x - ct))}{(\cosh(\alpha(x - ct)) + 2)^2} \tag{5.2}$$

where α is a constant and $c = -\alpha^2$. Explicitly, we have the equation

$$u_1^2 = \tfrac{8}{3}\sqrt{5}\alpha^3(a - u)^{3/2} + \tfrac{4}{15}(a - u)^3 - 4\alpha^2(a - u)^2 \tag{5.3}$$

where $a = 5\alpha^2$. Note that the right-hand side of the equation is analytic at at $u = 0$, and has a double zero there, but it is only C^1 at $u = a$.

This example shows that the equations for W can have branched solutions, while u is analytic. It is also interesting that the ansatz $u_1^2 = K(a - u)^{3/2} + Q(a - u)$, where Q is a polynomial with a double zero at $u = a$ actually singles out the Sawada-Kotera and Kaup equations, and leads to the solution above for the Kaup equation, and to a periodic solution of the Sawada-Kotera equation. But this latter one is a solution of the Sawada-Kotera equation when the integration constants are non-zero, hence it does not appear among the singular solutions above.

Acknowledgment

This work is partially supported by the Scientific and Technical Research Council of Turkey.

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